Parabola

Revision of concepts

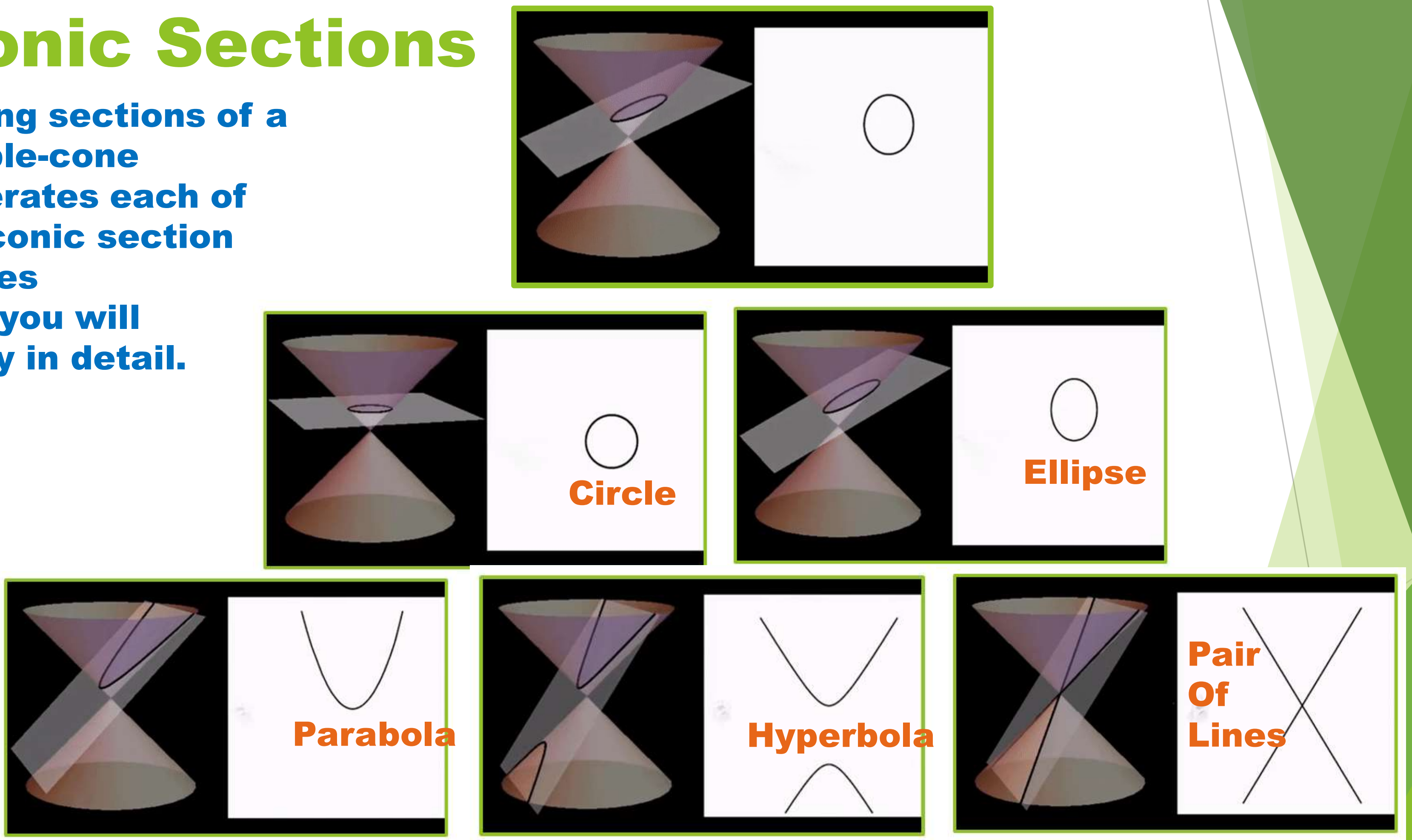
We will now revise:

- Conic sections
  - Common definition
  - Circle, Ellipse, Parabola, Hyperbola.
- Standard Equation of a parabola
- Geometry of a standard parabola
- Shifted parabola
- General equation of a parabola
- Position of a point w.r.t the parabola

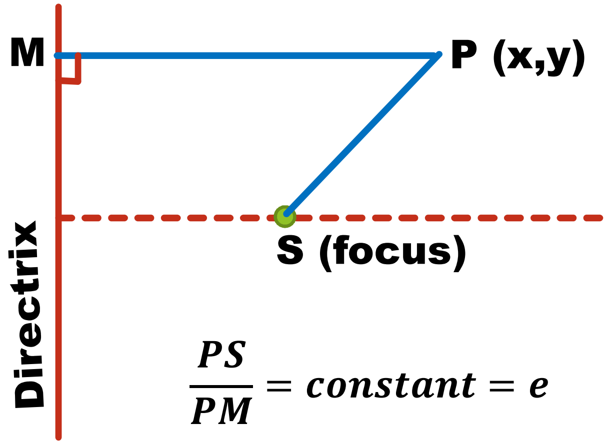
  and parametric point of a parabola.
- Tangents
  - Parametric form, Slope form, Point form
  - Properties
- Normals
  - Parametric form, Slope form, Point form
  - Geometric properties
  - Co-normal points
- Standard notations
  - Chord of contact
  - Pair of tangents
  - Chord with a given midpoint
- Other results and miscellaneous concepts to remember

Conic Sections

Taking sections of a double-cone generates each of the conic section curves that you will study in detail.



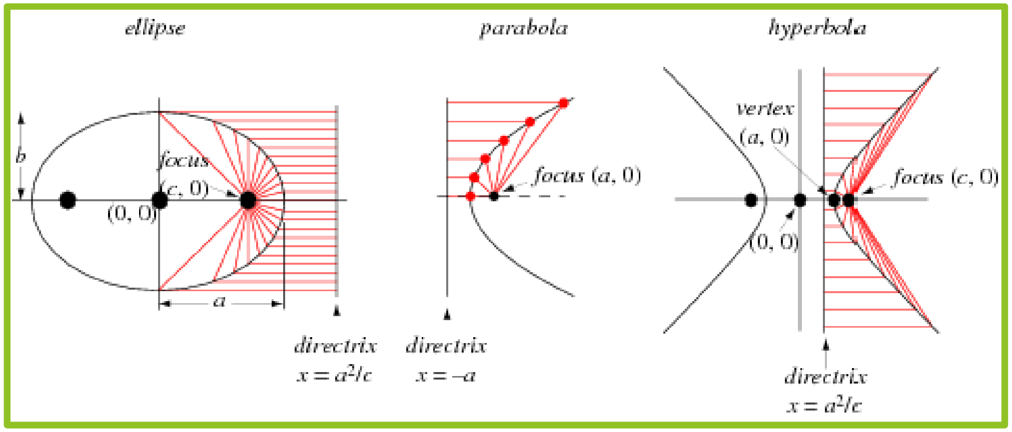
Conic Sections – Definition



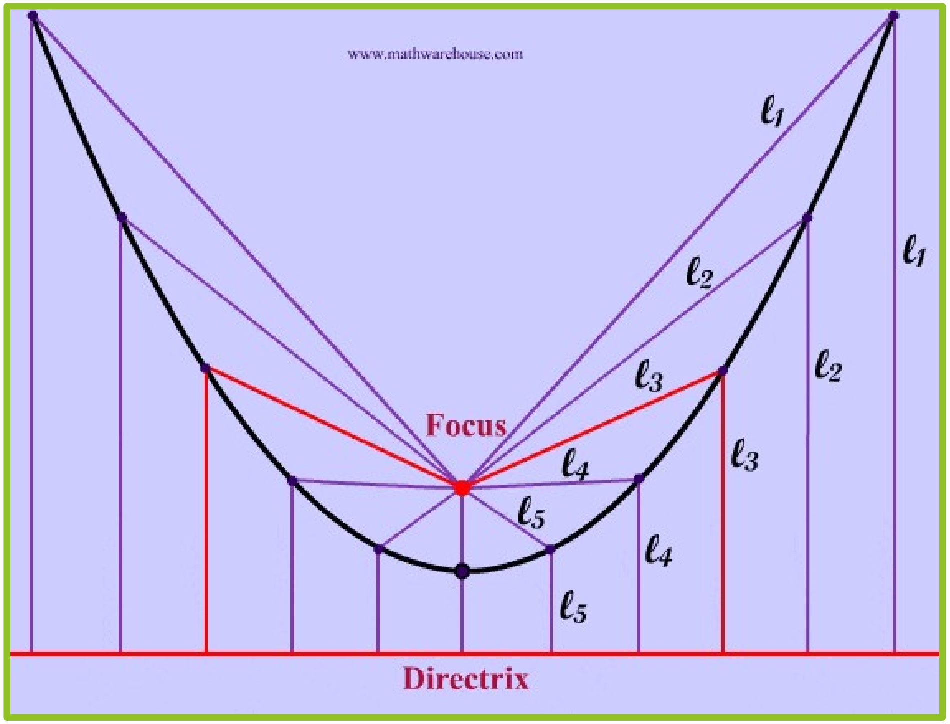
$$\frac{PS}{PM} = constant = e$$

P(x,y) is a variable point. It moves such that its distance from a fixed point (focus) and a fixed line (directrix) are at a constant ratio (eccentricity, e).

i. If e = 1, the conic is a parabola.
ii. If e < 1, the conic is an ellipse.
iii. If e > 1, the conic is a hyperbola.
iv. If e = 0, the conic is a circle.
v. If e = ∞, the conic is a pair of straight lines.



Parabola: e = 1



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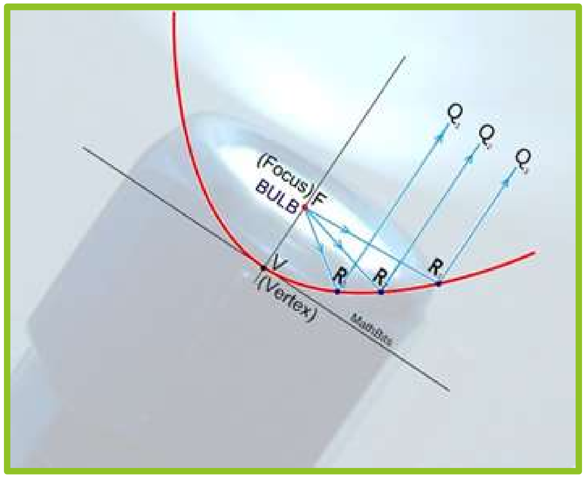
ℓ_1
ℓ_1
ℓ_2
ℓ_2
ℓ_3
ℓ_3
Focus
ℓ_4
ℓ_4
ℓ_5
ℓ_5
Directrix

Parabolas in real life

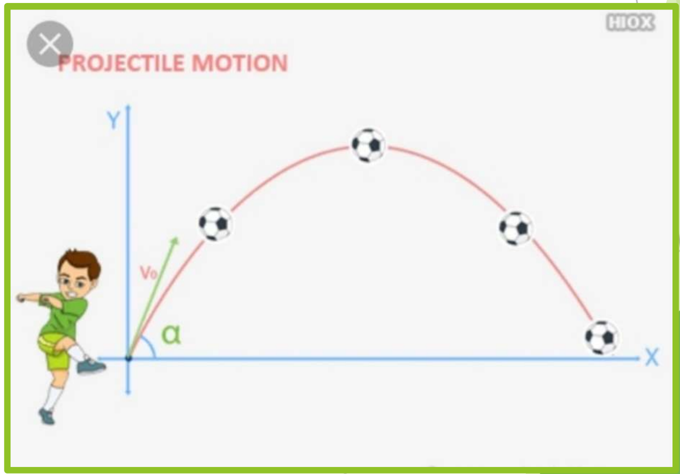
Suspension Bridges

Ski Jumping

Path of a diver

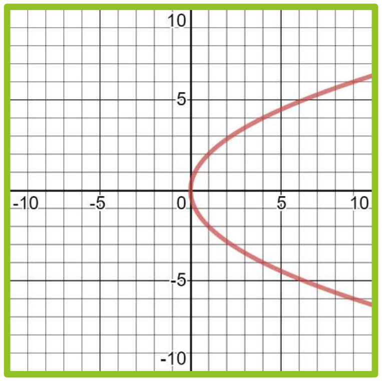


Reflectors

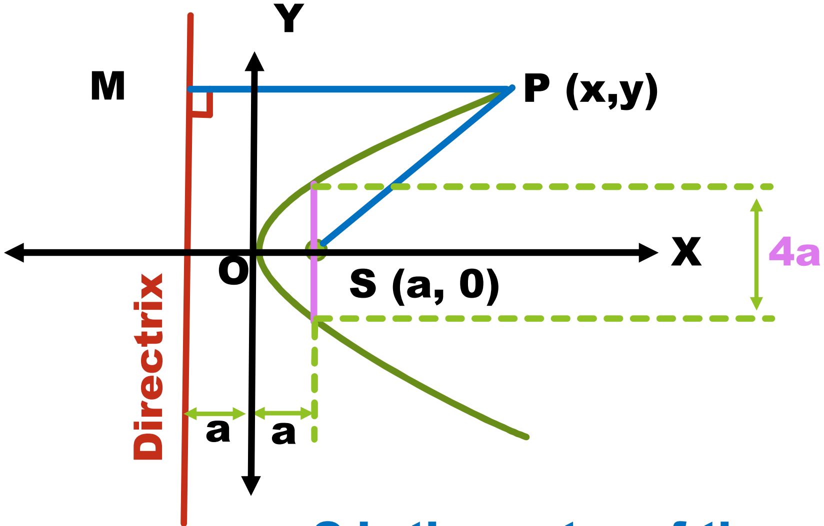


Sports

Equation and geometry of a standard parabola

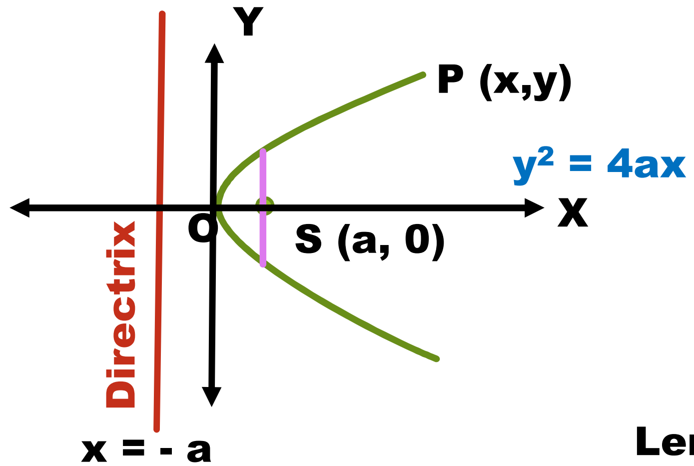


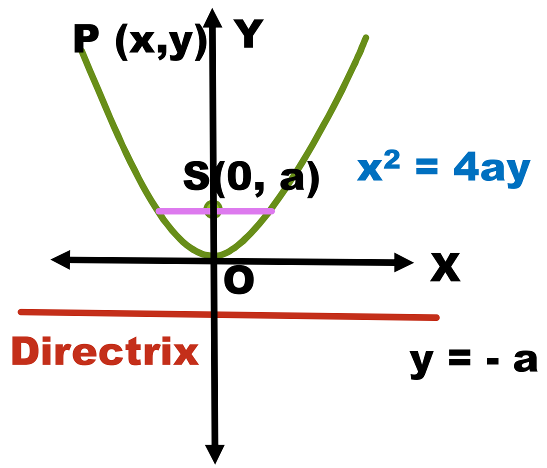
$y^2 = 4ax$



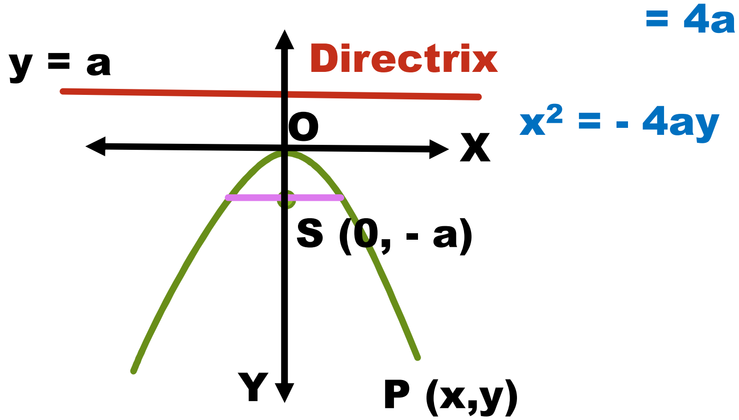
- O is the vertex of the parabola.
- S is the focus.
- x = - a is the directrix.
- x = 0 (Y - axis) is the tangent at the vertex
- Distance between focus and directrix = 2a
- Length of Latus rectum = 4a

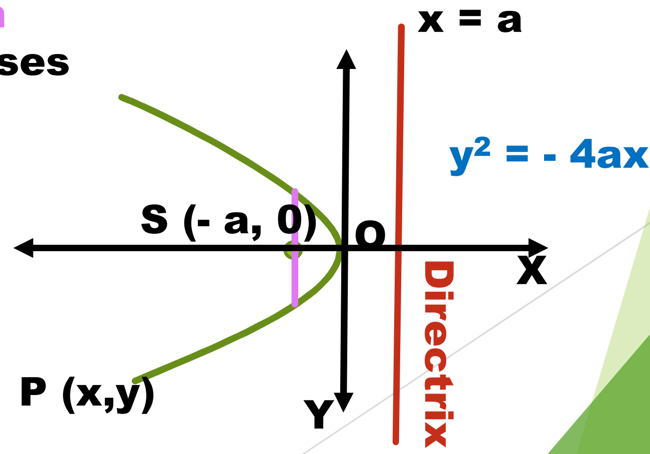
Other standard forms of parabola



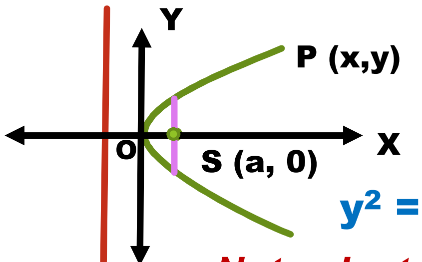


Length of
Latus rectum
= 4a in all cases

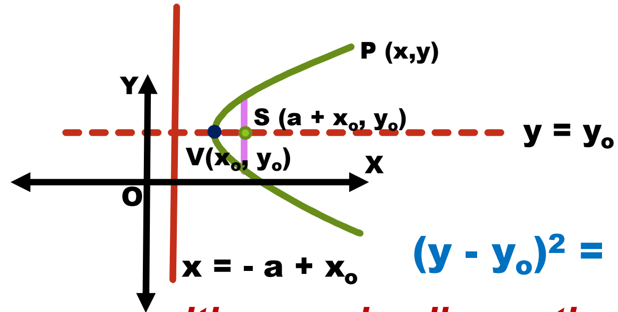




Shifted Parabola



$y^2 = 4ax$



$(y - y_o)^2 = 4a(x - x_o)$

Note: Just replace x with $x - x_o$, y with $y - y_o$ in all equations.
And add (x_o, y_o) in all coordinates.

- **Parabolas that have their vertex, V at a point other than the origin, and axis is parallel to X or Y axis.**

  $y^2 = 4ax$ ==► $(y - y_o) = 4a (x - x_o)^2$ … (1)

- **For (1), vertex is (x_o, y_o) and axis is parallel to X – axis.**
  - **Focus is (a, 0) + (x_o, y_o) i.e. $(a + x, y_o)$**
  - **Directrix is $x = - a + x_o$**
  - **All distances and geometric properties are same as that of $y^2 = 4ax$.**
- **When parabola equations are in the form of**
  - **$y = Ax^2 + Bx + C$ … (2) [or $x = Ay^2 + By + C$]**
- **Convert the equation to shifted parabola form by completing the square.**

General equation of a conic and identification of a parabola

- **Conics are represented by the general equation of second degree:**

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots (1)$$

- **Let** $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, **then** $\Delta = 0 \Rightarrow$ **(1) represents a pair of lines.**

- **When** $\Delta \neq 0$, **the nature of conics is as follows:**

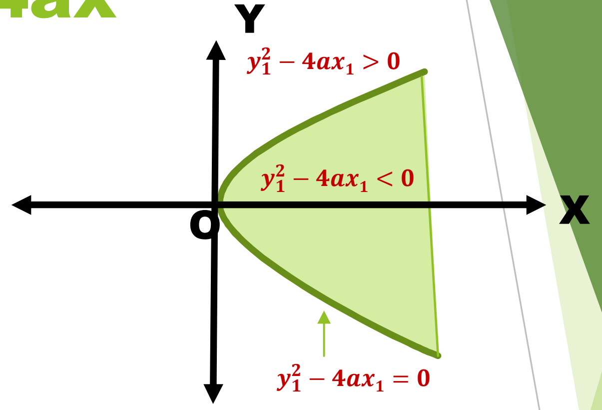
Condition	Nature of conics
$h = 0, a = b$	A circle
$h^2 = ab$	A parabola
$h^2 < ab$	An ellipse
$h^2 > ab$	A hyperbola
$h^2 > ab, a + b = 0$	A rectangular hyperbola

Position of a point w.r.t. $y^2 = 4ax$

- For a point $P(x_1, y_1)$, find the value of $y_1^2 - 4ax_1$
- If $y_1^2 - 4ax_1$ = 0, point is on the parabola.

  < 0, point is inside the parabola.

  > 0, point is outside the parabola.



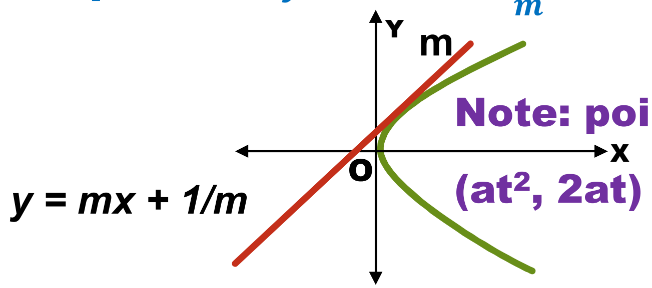
Parametric form of a point on the standard parabola

- Any point on $y^2 = 4ax$ can be represented by $(at^2, 2at)$
- Similarly,

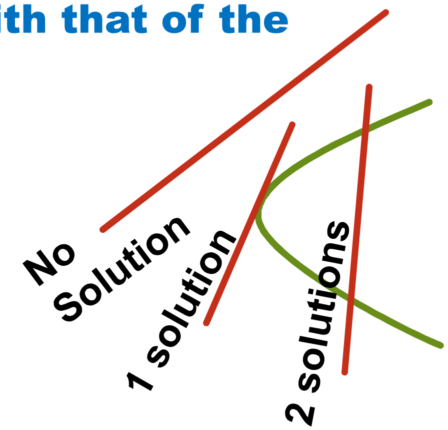
Parabola	Parametric point
$y^2 = -4ax$	$(-at^2, 2at)$
$x^2 = 4ay$	$(2at, at^2)$
$x^2 = -4ay$	$(2at, -at^2)$

Tangents to the parabola

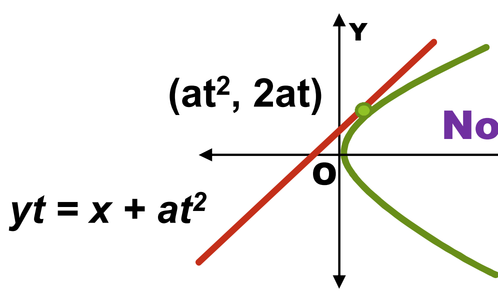
- **General form: Solve the equation of tangent with that of the parabola. 1 solution ⇒ tangent.**
- **Slope form:** $y = mx + \frac{a}{m}$



Note: point of contact is (at², 2at) = $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

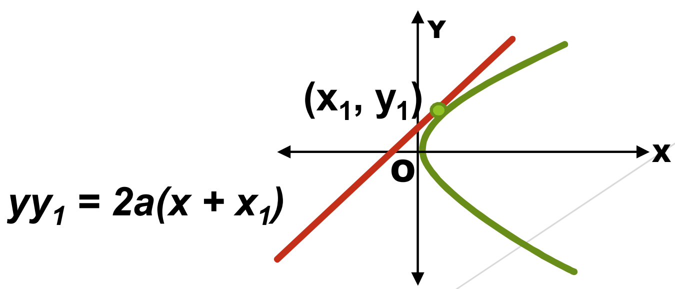


- **Parametric form:** $yt = x + at^2$



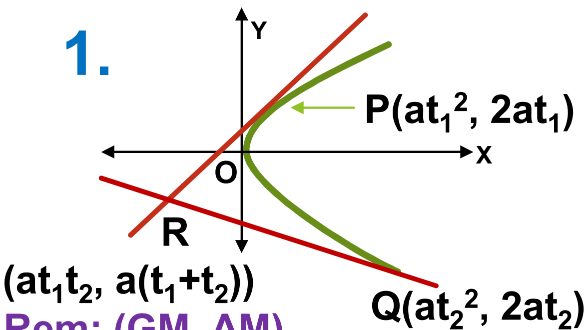
Note: m = $\frac{1}{t}$

- **Point form:** $yy_1 = 2a(x + x_1)$



Properties of tangents

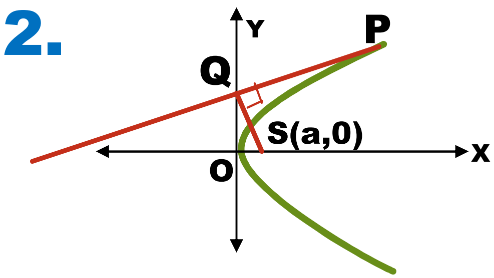
1.



Rem: (GM, AM)

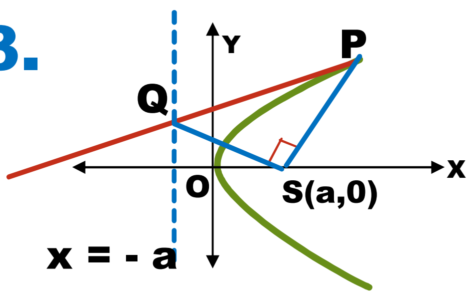
Point of intersection of two tangents.

2.



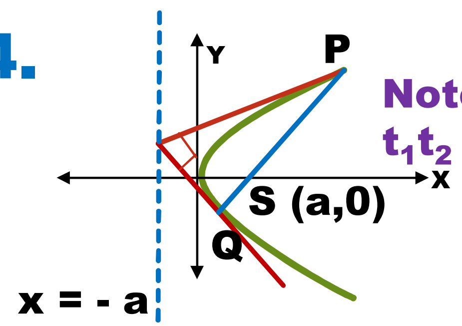
Foot of ⊥ from focus to tangent lies on tangent at vertex (i.e. Y- axis).

3.



Segment of tangent (at P) between P and directrix subtends 90° at focus.

4.

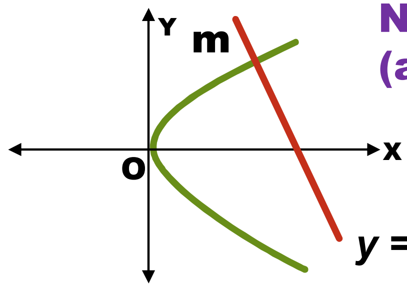


Note: For $P(t_1)$ and $Q(t_2)$, $t_1t_2 = -1$

Tangents at end points of focal chord are ⊥ and meet at the directrix.

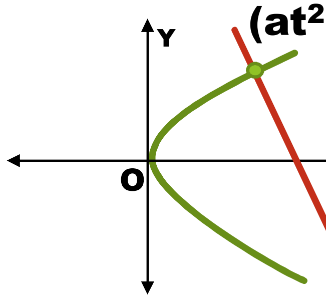
Normals

- **Slope form:** $y = mx - 2am - am^3$



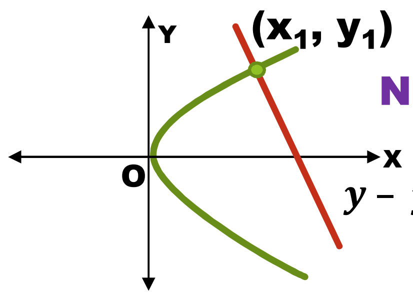
Note: foot of the normal is $(at^2, 2at) = (am^2, -2am)$

- **Parametric form:** $y = -xt + 2at + at^3$



Note: m = - t

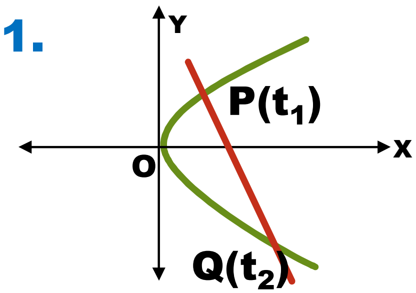
- **Point form:** $y - y_1 = -\frac{y_1}{2a}(x - x_1)$



Note: $m = -\frac{y_1}{2a}$

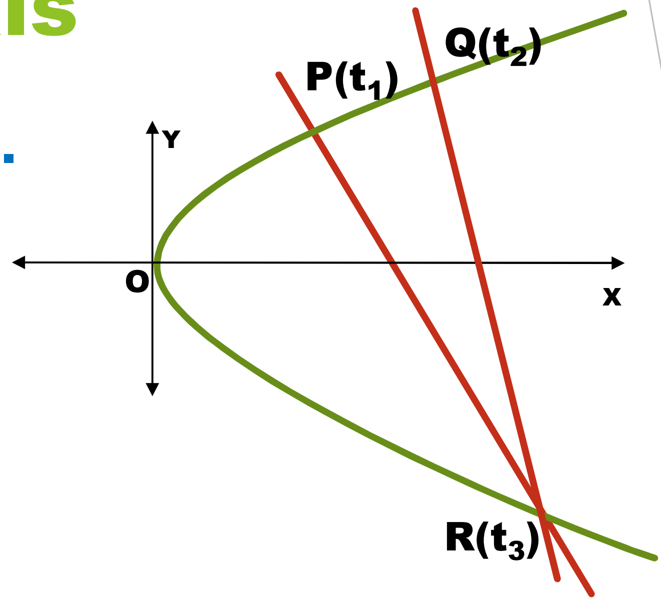
Properties of normals

1.



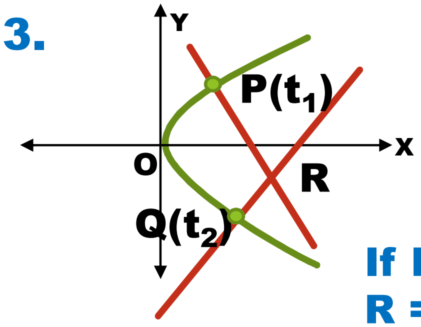
If normal at P(t_1) meets curve again at Q(t_2) then, $t_2 = -t_1 - \frac{2}{t_1}$

2.



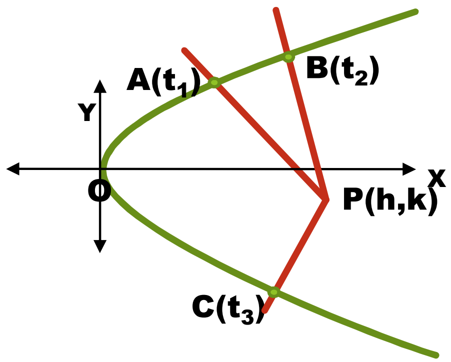
If normals at P(t_1) and Q(t_2) meet curve again at R(t_3) then, $t_1 t_2 = 2$ and $t_3 = -(t_1 + t_2)$
Also, PQ passes through fixed point (-2a,0)

3.



If Normals at P(t_1) and Q(t_2) meet at R, then
$R = (2a + a(t_1^2 + t_2^2 + t_1 t_2) , - a t_1 t_2 (t_1 + t_2))$

Co-normal points and their properties



If from P(h,k) three normals are drawn to $y^2 = 4ax$, then,

$m_1 + m_2 + m_3 = 0 = t_1 + t_2 + t_3$

$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a} = t_1t_2 + t_2t_3 + t_3t_1$

$m_1m_2m_3 = -t_1t_2t_3 = \frac{-k}{a}$

Properties:

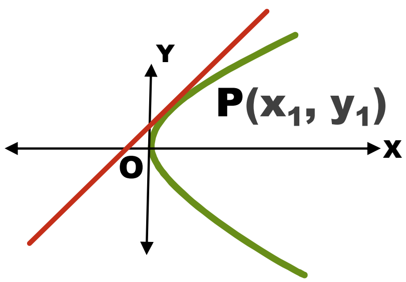
1. The centroid of Δ ABC lies on the axis of the parabola.
2. Three normals from P(h,k) are possible only if $h > 2a$.
3. Algebraic sum of slopes of normals is zero.
4. Algebraic sum of ordinates of A, B, C is zero.
5. Circle through A, B, C passes through the origin.

Standard Notations:

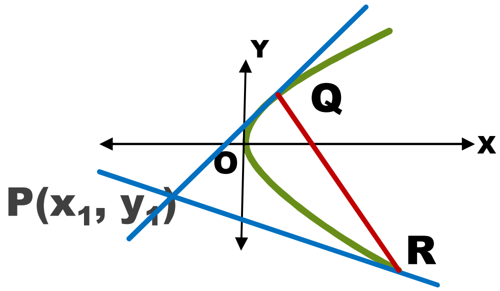
- Let $S \equiv y^2 - 4ax$, $S_1 \equiv yy_1 - 2a(x + x_1)$, $S_{11} \equiv y_1^2 - 4ax_1$

  Then,

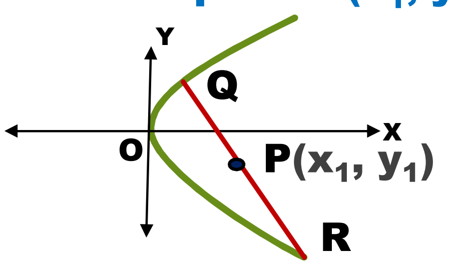
- Equation of tangent at $P(x_1, y_1)$ on the parabola is $S_1 = 0$



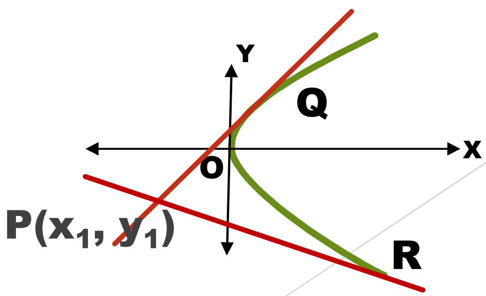
- Equation of chord of contact, QR of external point $P(x_1, y_1)$ is $S_1 = 0$



- Equation of chord PQ with given mid point $P(x_1, y_1)$ is $S_1 = S_{11}$



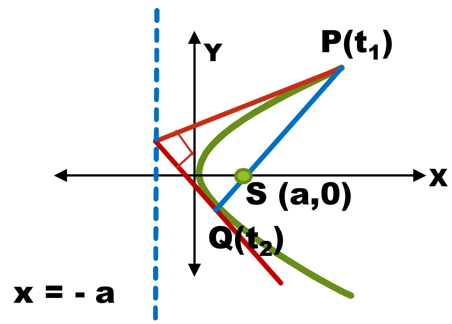
- Equation of pair of tangents PQ, PR from $P(x_1, y_1)$ is $S.S_{11} = S_1^2$



Other useful results to remember

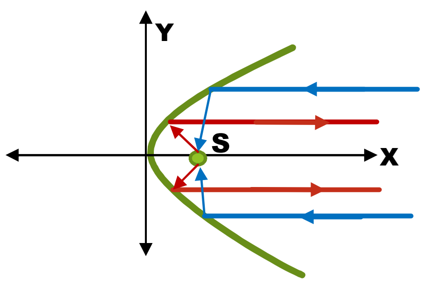
- If $P(t_1)$ and $Q(t_2)$ are end points of a focal chord of $y^2 = 4ax$ then, $t_1 t_2 = -1$,

  tangents at P and Q are $\perp$ and meet at the directrix.



- Reflection property of parabola:

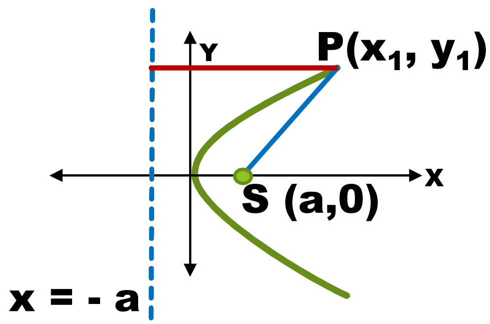
  If a ray of light is sent along a line parallel to the axis of the parabola, then the reflected ray passes through the focus and vice versa.



Other useful results to remember

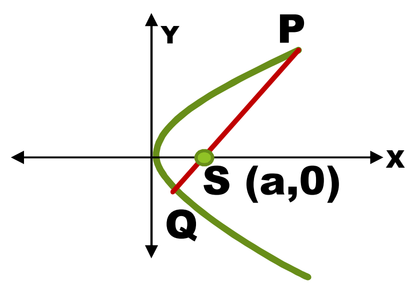
- If $P(x_1, y_1)$ is a point on $y^2 = 4ax$, and S (a, 0) is the focus, then

  $PS = x_1 + a = a(t^2 + 1)$ in parametric form.



- If PQ is a focal chord, S is the focus, then semi-Latus Rectum is Harmonic Mean of PS and QS.

$$\frac{1}{PS} + \frac{1}{QS} = \frac{1}{a} = \frac{2}{SLR}$$



Miscellaneous Concept

- **Special form of non-standard parabola:**

  An equation of the form $(ax + by + c)^2 = \lambda (bx - ay + d)$ ... (1)

  represents a parabola with axis $ax + by + c = 0$, and tangent at vertex being $bx - ay + d = 0$.

  (Hint: The two lines are clearly $\perp$ and (1) resembles $Y^2 = 4aX$ where Y and X are distances from the set of $\perp$ lines.)

Eg. $(x-y)^2 = 2(x+y-1)$ represents a parabola with axis $x - y = 0$, tangent at vertex $x + y - 1 = 0$. Vertex is point of intersection of the two lines.

To identify focus, length of latus rectum etc, write the above equation as $D_1^2 = 4pD_2$ where D_1 and D_2 are distances of (x,y) from the two lines. Then apply geometry of parabolas.

The End ☺